Worksheet- Set theory **Name:**
Part 3: 12/10/2024

Warm-up: There are 29 students in a class. 15 students swim after school and 21 play football. 3 students do not do any of these sports. How many kids both swim and play football?

1. (a) Create 5 examples of functions you can identify from our daily lives. How many of them are injective/surjective/bijective? Make the examples as interesting as possible!
   (b) Can you find a function whose graph passes through the points $(0,0), (1,2), (1,4)$, and $(2,4)$?
   (c) A car travels at a constant speed of 40 km/h. Write down a function that tells you the distance traveled given the total travel time.
   (d) The IIT Palakkad library charges Rs. 5/- per day for an overdue book. Write a function that takes as input "the no of days overdue" and gives as output the amount to be paid as a fine.
   (e) The library decided to charge 10 rupees for every financial transaction. Can you modify the function you constructed to reflect this change?
   (f) Construct a function whose graph passes through the points $(0,0)$ and $(1,1)$. Can you construct another function that passes through the same points
   (g) Construct a polynomial $f : \mathbb{R} \to \mathbb{R}$ which takes the value 0 at $x = 1, 2$ and 3.
   (h) Construct a polynomial $f : \mathbb{R} \to \mathbb{R}$ which takes the value 0 at $x_1, x_2, x_3 \cdots x_k$.
2. Check if the following two functions are equal
   (a) Consider the functions $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = g(x) = x^2$.
   (b) Let $X = \{-1, 0, 1\}$ and $f, g : X \to \mathbb{R}$ be defined by $f(x) = x$ and $g(x) = x^3$ for $x \in X$.
3. Let $f, g : X \to Y$. If $Range(f) = Range(g)$. Can we conclude $f = g$?
4. The maximal domain of a function is defined as the largest subset of $\mathbb{R}$ on which it is well-defined. Consider $f(x) = \sqrt{x}$, $g(x) = (x+1)^2$ as functions defined on their maximal domains. Write down the following functions in case they are well-defined: $f \circ f$, $f \circ g$, $g \circ f$, $g \circ g$.
5. Sketch, using translation and change of scale (express them as compositions of simpler functions). Are any of these functions injective/surjective/bijective?
   (a) $y = x^2 - 2x - 1$
   (b) $y = 3x^2 + 6x + 2$
   (c) $y = 1 + |x + 2|$
   (d) $y = \frac{10}{(x-1)^3}$
6. Which of the following describes the graph of a function? If it is describing a function, state it explicitly.
   (a) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$
   (b) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1 \text{ and } y \geq 0\}$
   (c) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x = |y|\}$
   (d) $\{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + 4y = 0\}$

7. An even function is a real function such that $f(-x) = f(x)$ for every x in its domain. Similarly, an odd function is a real function such that $f(-x) = -f(x)$ for every x in its domain. Identify each of the following as even, odd, or neither. Let $E(x)$ denote an arbitrary even function and $O(x)$ denote an arbitrary odd function.

   (a) $x^2 + x^6 + 5$
   (b) $(x-1)^2$
   (c) x^3
   (d) $x^3 + 1$
   (e) $x^3 + 3x$
   (f) $(x-1)^3$
   (g) $\sin^2 x$
   (h) $\frac{\tan x}{1+x^2}$
   (i) $O \circ E$
   (j) $E \circ O$

8. A function $f(x)$ is said to be periodic with period P if $f(x) = f(x+P)$. Informally, the function repeats itself in regular intervals of length P. Suppose $f(x)$ is odd and periodic. Show that the graph of $f(x)$ crosses the x-axis infinitely often.

9. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a polynomial of degree n if it can be expressed in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \cdots, a_0$ are constants called the coefficients of the polynomial. Write down examples of polynomials. Show that every polynomial is the sum of an even and an odd function.

10.* Can you write any function as the sum of an even and an odd function?

11.* Construct a polynomial $f : \mathbb{R} \to \mathbb{R}$ which takes the value 1 at $x = 3$ and the value 0 at $x = 1, 2$ and 4.

12.* Graph the function f that consists of straight line segments joining the points $(-1, -1)$, $(1, 2)$, $(3, -1)$, and $(5, 2)$. Such a function is called piecewise linear.

   (a) Extend the graph of f periodically. What is its period?
   (b) Graph the function $g(x) = 3f((x/2) - 1) - 3$