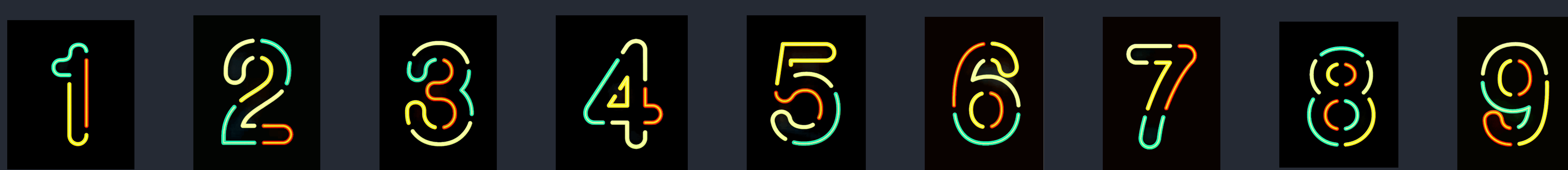
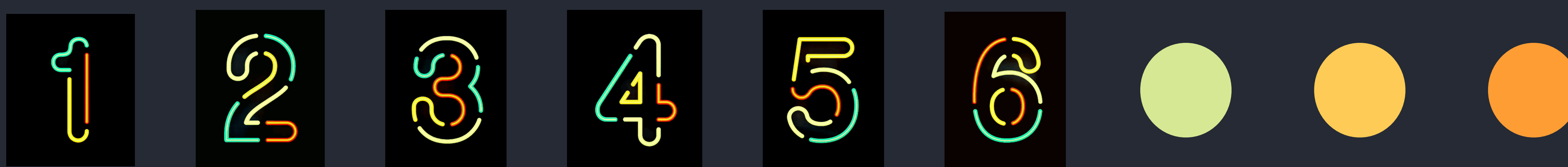
AN INTRODUCTION TO SET THEORY



WHAT IS A SET?

- Mathematical objects - can be formed into collections
- Natural numbers

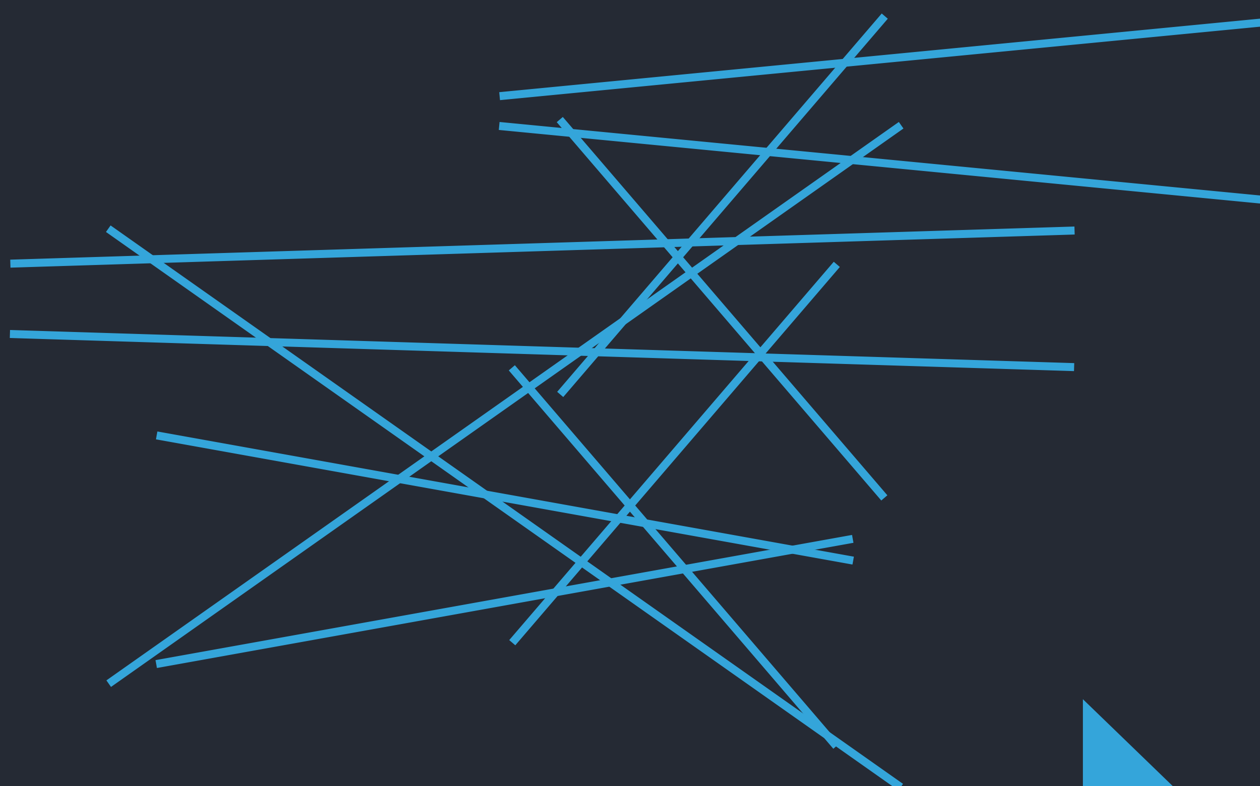


- Even natural numbers

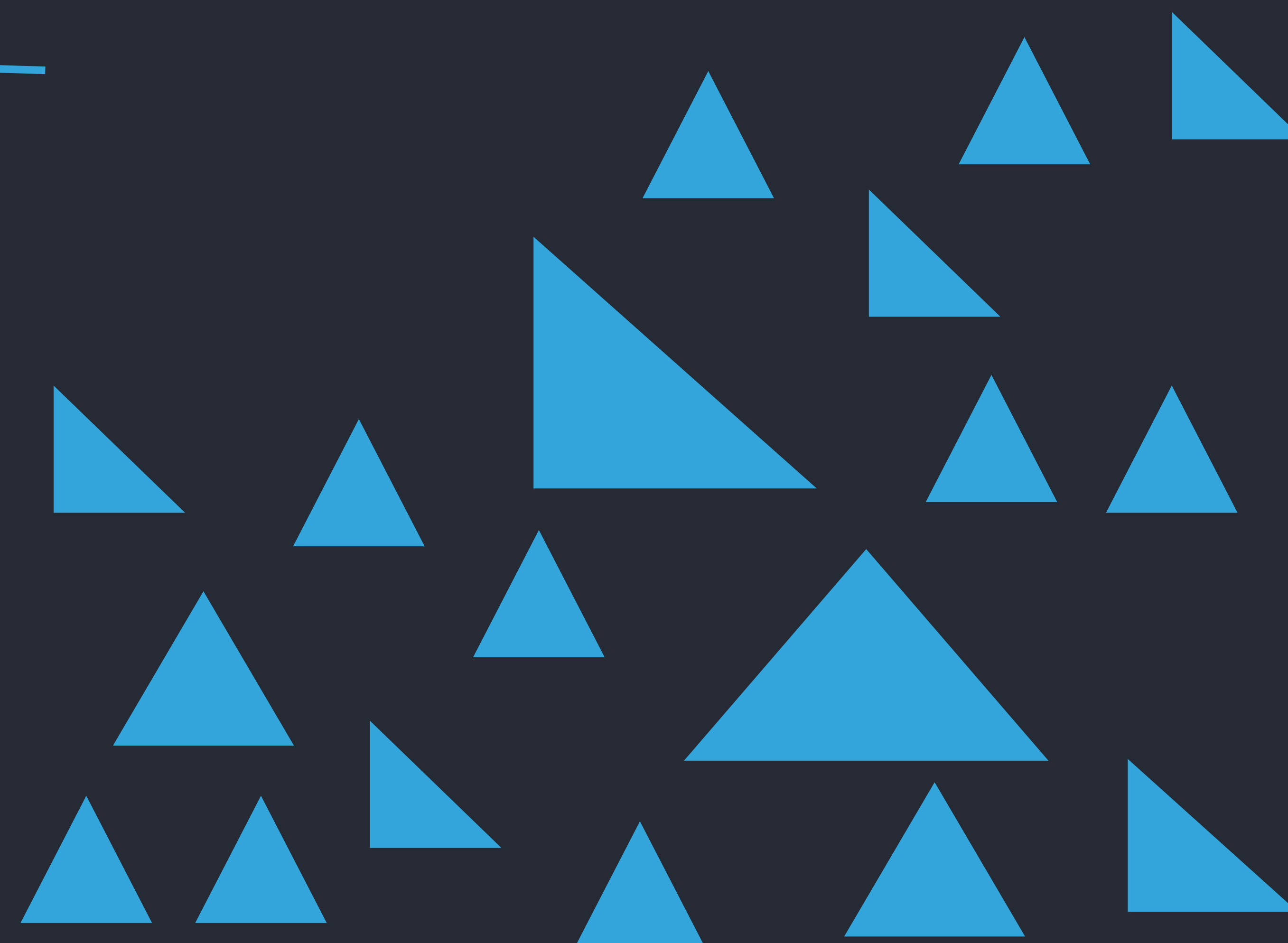
- Odd natural numbers

WHAT IS A SET?

- Lines on a plane



- Triangles on a plane



- Typically objects are studied collectively as a group

WHAT IS A SET?

- A set is a well-determined collection of distinct objects
  - All prime numbers
  - All IITPkd math circle students
  - $\diamondsuit, \heartsuit, \spadesuit, \clubsuit$
  - $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$
  - Hello, hi, bye
- Well-determined refers to a specific property which makes it possible to identify whether a given object belongs to a set or not

NOTATION

- Let S be the set of all even integers from 1 to 10

$$S := \{2,4,6,8,10\}$$

- Ordering or multiplicity of elements is not relevant

$$\{4,2,6,10,8\} \qquad \{4,2,6,10,8,2,4\}$$

- An object is a member of a set if it is one of the objects in the set

- 6 is an element of S

$$6 \in S$$

- 5 is not an element of S

$$5 \notin S$$

NOTATION

- Another notation for a set with many elements following implicit pattern

$$X := \{1,2,3,\ldots,10\}$$

- Which set does the following denote?

$$Y := \{3,5,7,\ldots\}$$

  - The set of all odd numbers greater than 1?
  - The set of all odd prime numbers?
  - Not well-determined!

NOTATION

- Set-builder notation: a way of describing a set by stating the properties that its members must satisfy

$$S := \{x \in \mathbb{N} : x \text{ is even}\}$$

$$T := \{x : x \text{ is non-negative and even}\}$$

$$P := \{x : x \text{ is prime}\}$$

- The empty set has no elements $\varnothing$ $\{\,\}$

RELATIONSHIPS BETWEEN SETS

$X := \{2,3,6\}, Y := \{1,2,3,6\}$

- Equality $X = Y$
- Subset $X \subseteq Y$
- Strict Subset or Proper Subset $X \subsetneq Y$
- Superset $Y \supseteq X$
- Strict Superset or Proper Superset $Y \supsetneq X$
- The power set of a set X is the set of all subsets of X

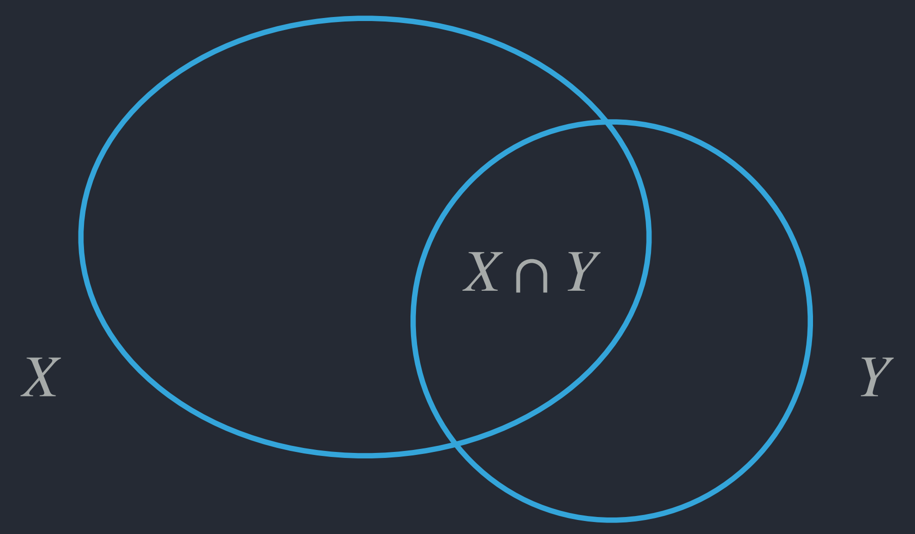
OPERATIONS ON SETS

- Union of two sets

$$X \cup Y := \{z : z \in X \text{ or } z \in Y\}$$

- Intersection of two sets

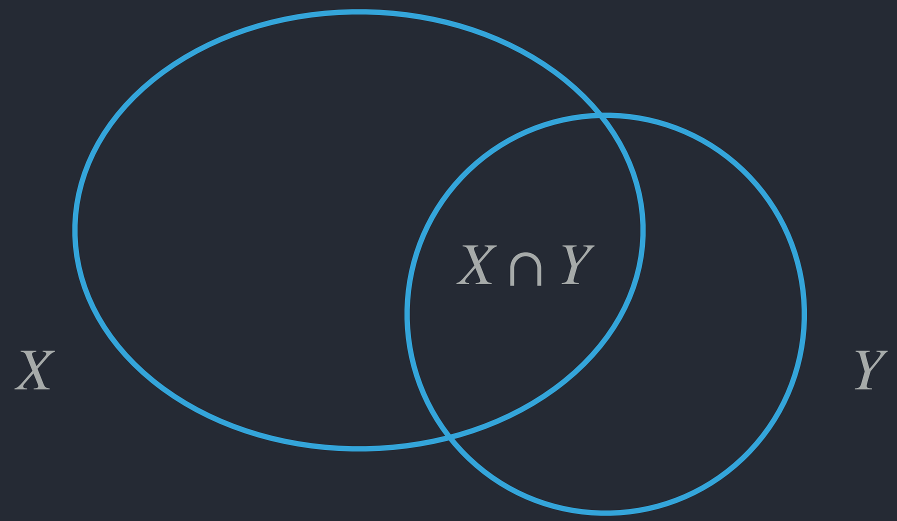
$$X \cap Y := \{z : z \in X \text{ and } z \in Y\}$$



RELATIONSHIPS BETWEEN SETS BASED ON INTERSECTION

- X and Y are said to be disjoint if $X \cap Y = \emptyset$

- X and Y are said to overlap if $X \cap Y \neq \emptyset$



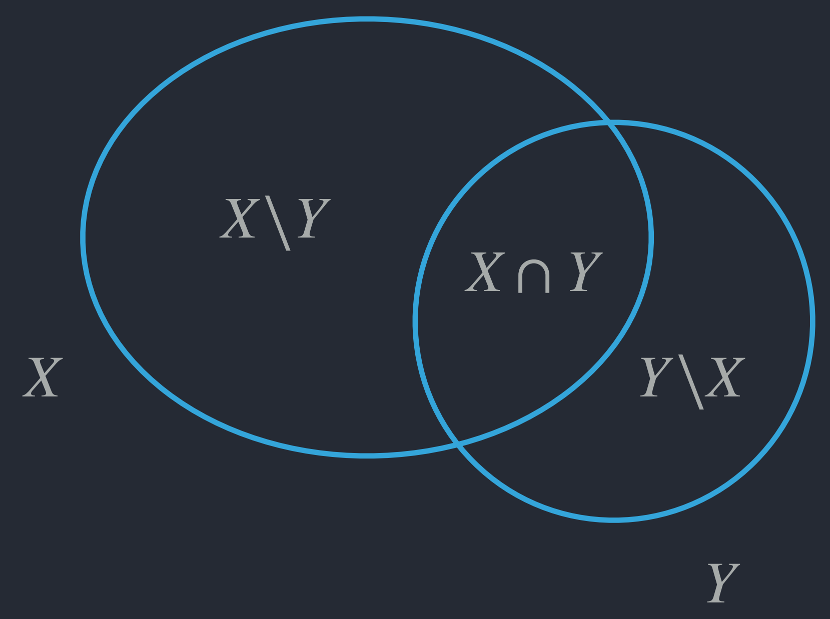
OPERATIONS ON SETS

- Difference of two sets

$$X \setminus Y := \{z : z \in X \text{ and } z \notin Y\}$$

- Symmetric difference of two sets

$$X \Delta Y := (X \setminus Y) \cup (Y \setminus X)$$



OPERATIONS ON SETS

- Universe U of all elements under consideration
- Given $X \subseteq U$, the complement of X denoted by $\overline{X}$ or X^c is $U \backslash X$

- Let U be the set of integers and A be the set of odd integers. Then $\overline{A}$ is the set of even integers.
- Let U be the set of integers and B be the set of multiples of 3. Then $\overline{B}$ is the set of integers that are not multiples of 3.

OPERATIONS ON SETS

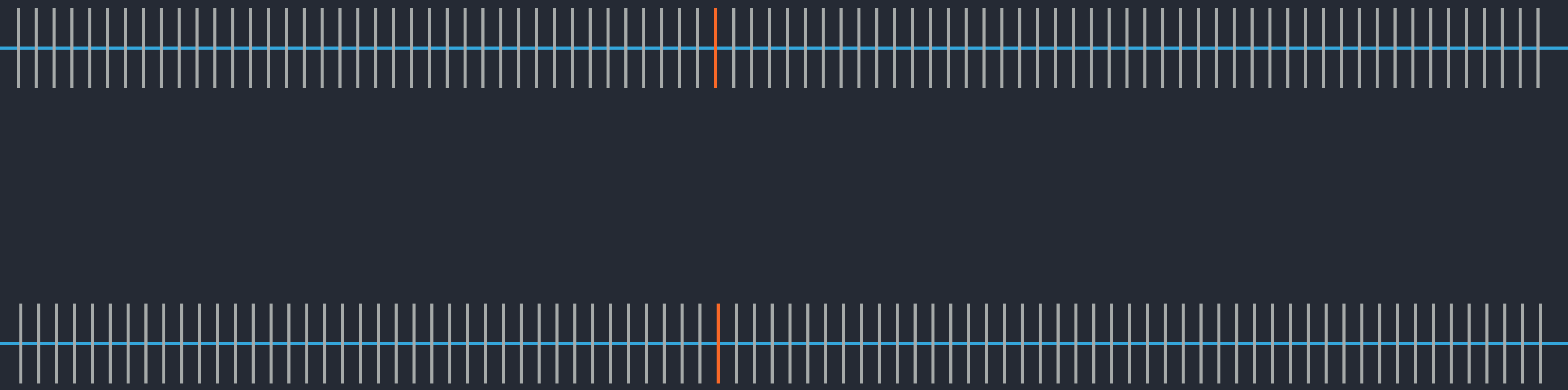
- Cartesian product of X and Y: the set of all ordered pairs (a, b) where a is in X and b is in Y

$$X \times Y := \{(a, b) : a \in X, b \in Y\}$$

$$\left\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\right\} \times \left\{\blacklozenge, \heartsuit, \spadesuit, \clubsuit\right\}$$

$$= \left\{(2, \blacklozenge), (3, \blacklozenge), (4, \blacklozenge), \ldots, (A, \blacklozenge), (2, \heartsuit), \ldots, (A, \clubsuit)\right\}$$

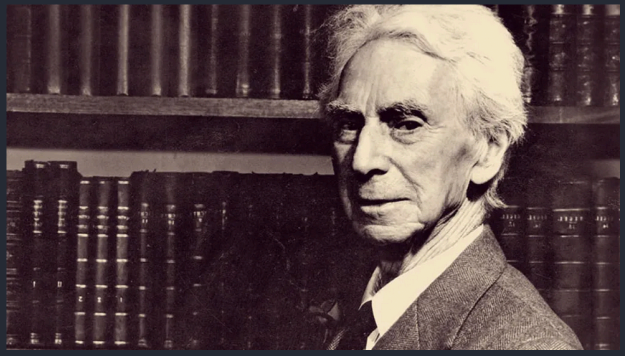
$$\mathbb{R} \times \mathbb{R} := \{(x, y) : x, y \in \mathbb{R}\}$$



$\mathbb{R}^2$

RUSSEL'S PARADOX

- A set is a well-determined collection of distinct objects
  - Object: undefined, Collection: undefined
  - Well-determined: refers to a specific property which makes it possible to identify whether an object belongs to the set or not
- Call a set X ordinary if $X \notin X$ and extraordinary otherwise
- Can you give examples of extraordinary sets?
- Every set is either ordinary or extraordinary
- Let $\mathcal{O} := \{X : X \notin X\}$ be the set of all ordinary sets.
- Is $\mathcal{O}$ an ordinary set?



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DEFINITION OF A SET

- We need a statement of the conditions under which sets are formed
- Problem with the naive definition: for any property there exists a set whose members are precisely those objects that satisfy the property
- Zermelo-Fraenkel set theory with choice (ZFC)
  - Does not allow a set corresponding to every property
  - Does not allow a set to contain itself
  - Does not allow a set containing all sets
- A set is a well-determined collection of distinct objects that satisfies the ZFC conditions