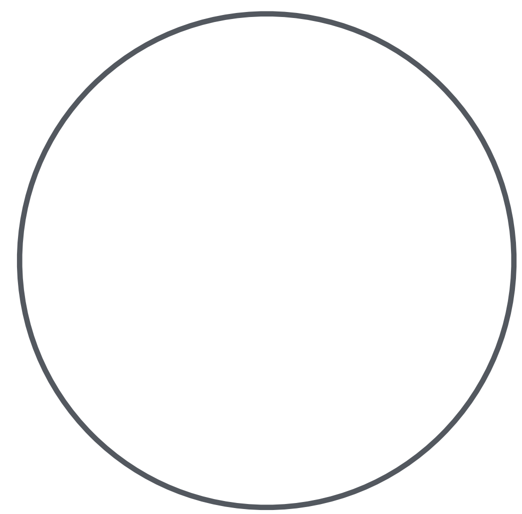
SEQUENCES & SERIES

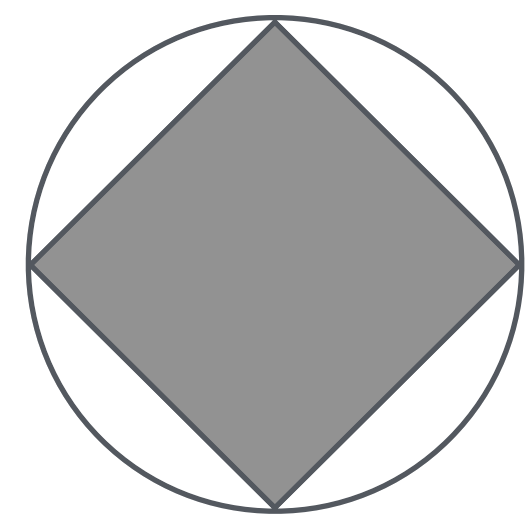
A gentle introduction to limits

Palakkad Math Circle | 13 Sep 2025

APPROXIMATING π USING INSCRIBED POLYGONS

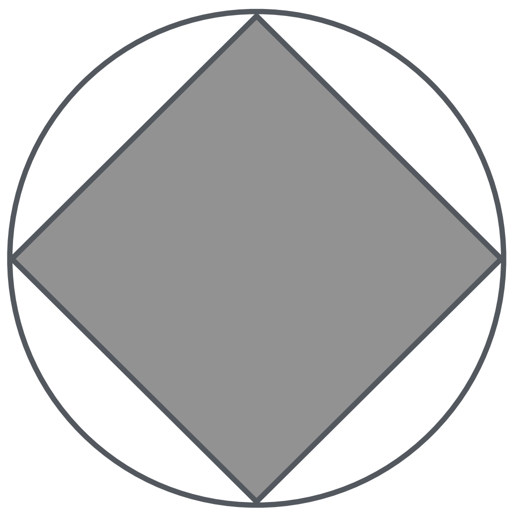
- Consider a unit circle and let π denote its area. How can we estimate π?



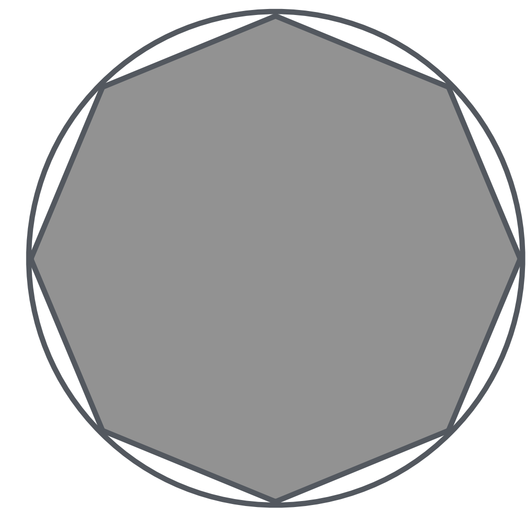


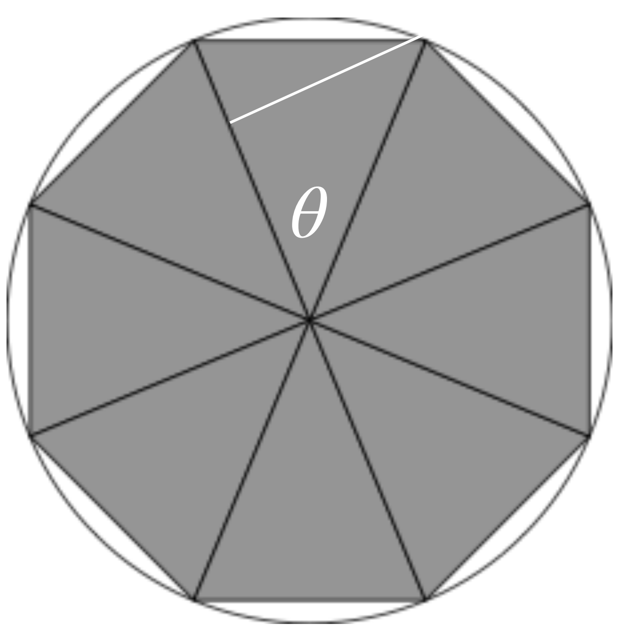
- Inscribe a square in this circle. <u>How?</u>
- Area of the square inscribed in this circle is less than the area of the circle
  - What is the area of this square?
- Is the area of this square a good estimate for π?

APPROXIMATING π USING INSCRIBED POLYGONS



- By bisecting each side of the square, we get the vertices of a regular octagon inscribed within the circle

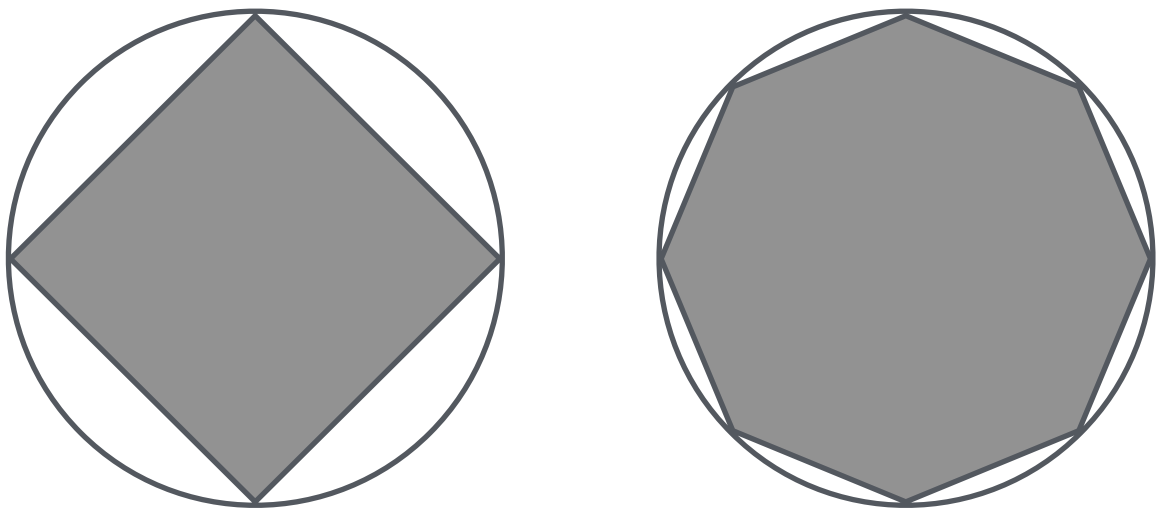




- Divide the octagon into 8 isosceles triangles - each with central angle $\theta = 2\pi/8 = \pi/4$
- Area of one triangle $= ?$
- Area of the octagon $= ?$

- Is the area of this octagon a good estimate for π?

APPROXIMATING π USING INSCRIBED POLYGONS



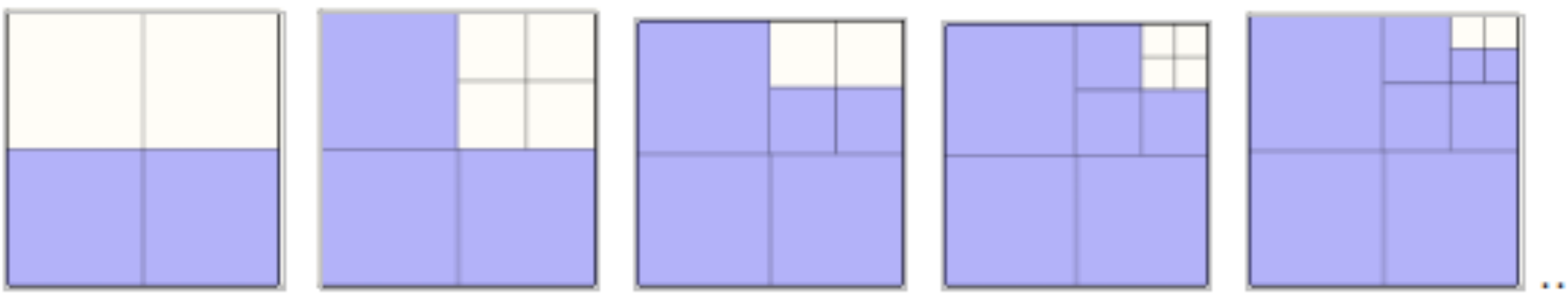
. . .

This process can be continued indefinitely - an infinite process!

- Bisect each side of the octagon to get a regular 16-gon and continue …
- Area of the n-sided regular polygon inscribed in the unit circle is $\frac{n}{2}\sin(\frac{2\pi}{n})$
- As n increases, the area of the regular polygon with n sides inscribed in the circle gets closer and closer to <u>the area of the circle</u>

ANOTHER INFINITE PROCESS

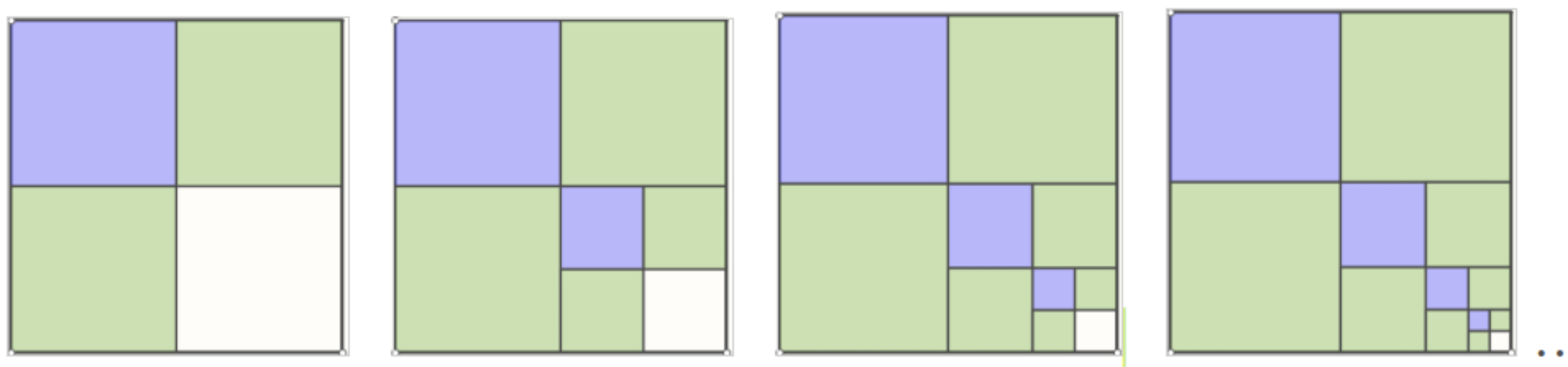
- Consider an unit square



- What is the area of the purple region?
- $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots$
- "Eventually", the purple portions covers the square
- $\frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots$ approaches 1

ANOTHER INFINITE PROCESS

- Consider an unit square



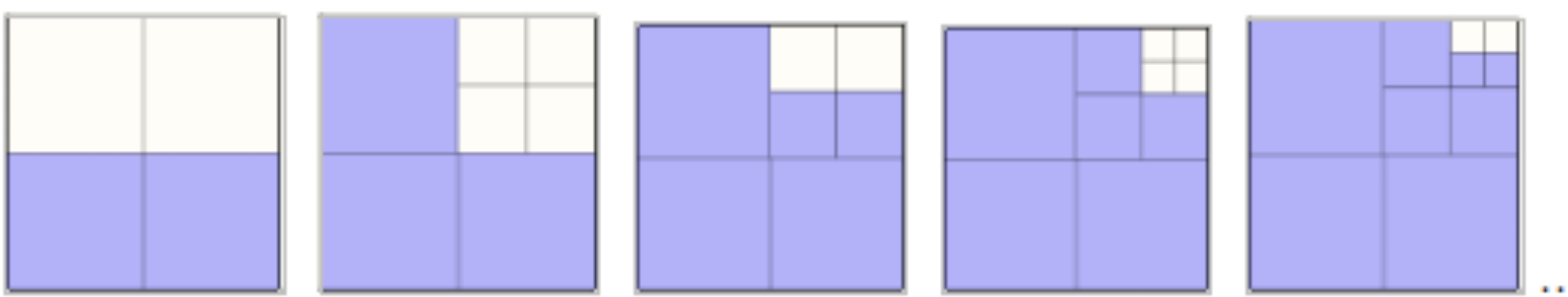
- What is the area of the purple region?

Image courtesy: Prof, John Mason, Open University, UK

ANOTHER INFINITE PROCESS

- Chocolate in a Box
  - Save the boxes, get 1 chocolate box in exchange of 10 empty boxes
  - What is the worth of an empty box in terms of chocolate units?

SEQUENCES

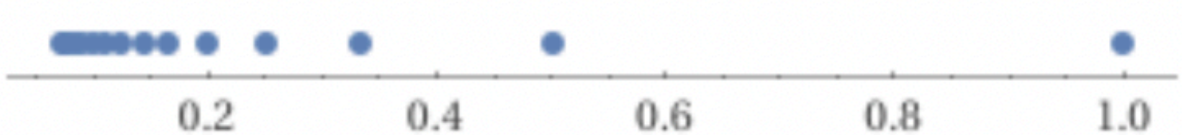
- Numerical sequence: an ordered list of real numbers $x_1, \; x_2, \; x_3, \; \ldots$
  - Eg: 1, 2, 3, 4, 5, …. notation: (n)
- Approximating π
  - Sequence of areas: $A_4, \; A_8, \; A_{16}, \; A_{32}, \; \ldots$
- 

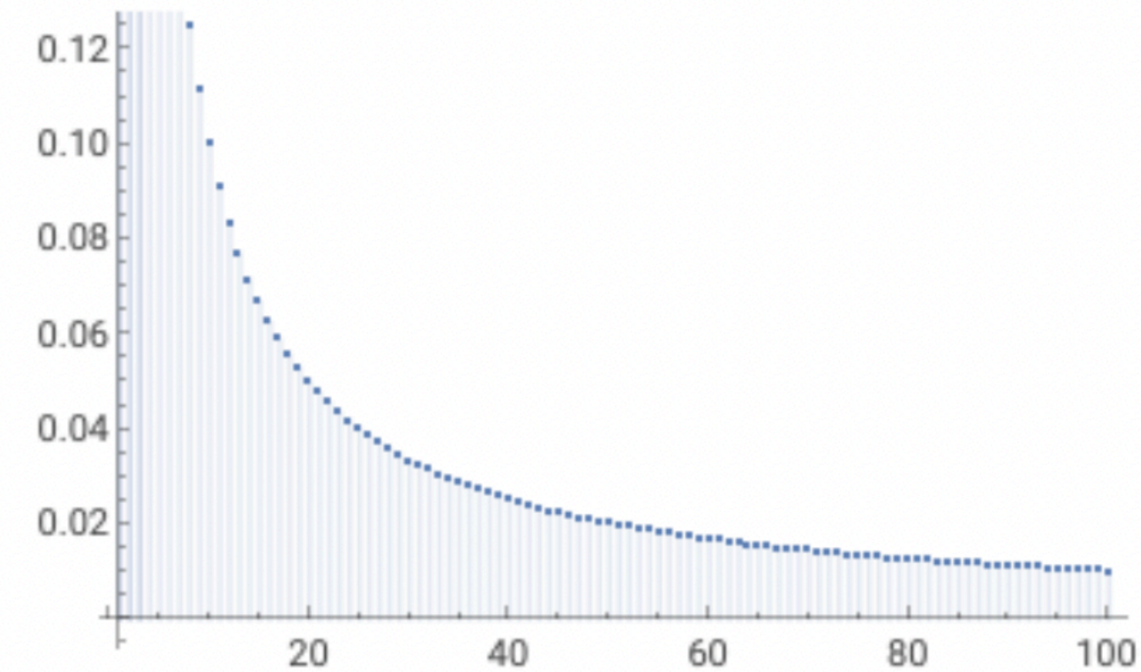
$\frac{1}{2}$ $\qquad \frac{1}{2} + (\frac{1}{2})^2$ $\qquad \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3$ $\qquad \cdots$

SEQUENCES

- $(1/n) = 1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$

Number line





Decimal approximation

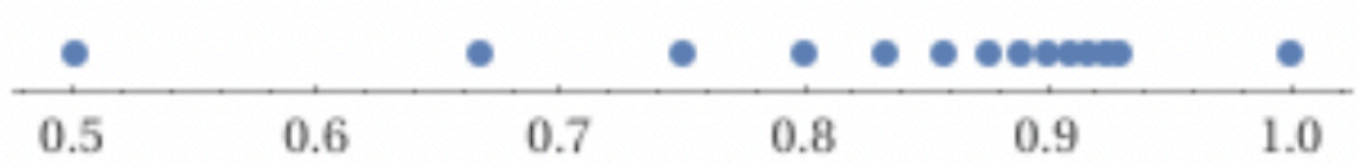
{1, 0.5, 0.333333, 0.25, 0.2, 0.166667, 0.142857, 0.125, 0.111111, 0.1, 0.0909091, 0.0833333, 0.0769231, 0.0714286}

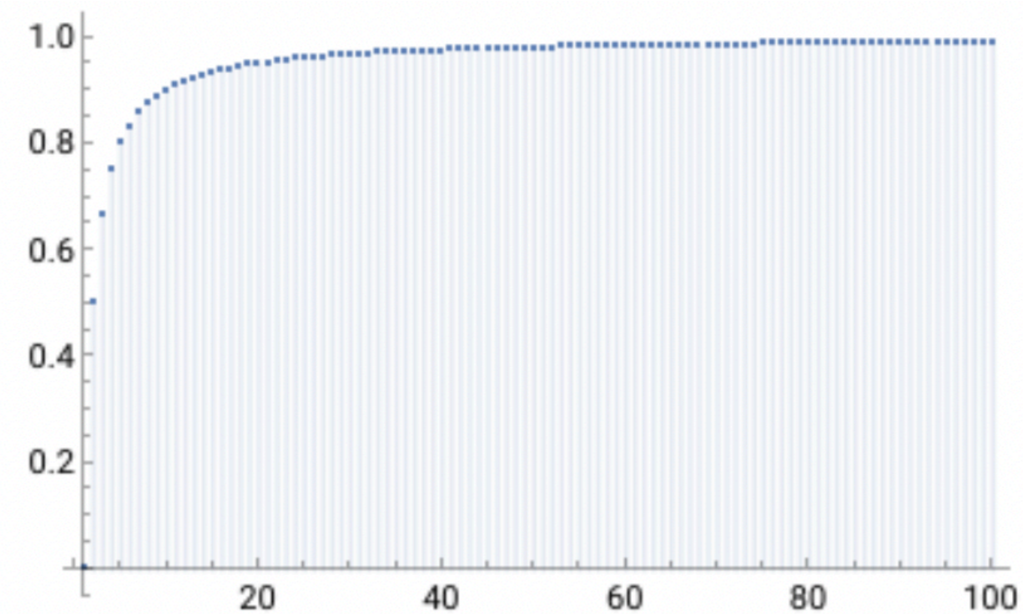
- Decreasing, "approaching 0". How to formalise $(1/n)$ "approaches 0"?
    - As n increases, terms get closer and closer to 0 (terms accumulate near 0)

SEQUENCES

- $(1 - \frac{1}{n}) = 0,\ 1/2,\ 2/3,\ 3/4,\ \ldots$

Number line





Decimal approximation

{1, 0.5, 0.666667, 0.75, 0.8, 0.833333, 0.857143, 0.875, 0.888889, 0.9, 0.909091, 0.916667, 0.923077, 0.928571}

- Increasing, "approaching 1"
  - As n increases, terms get closer and closer to 1

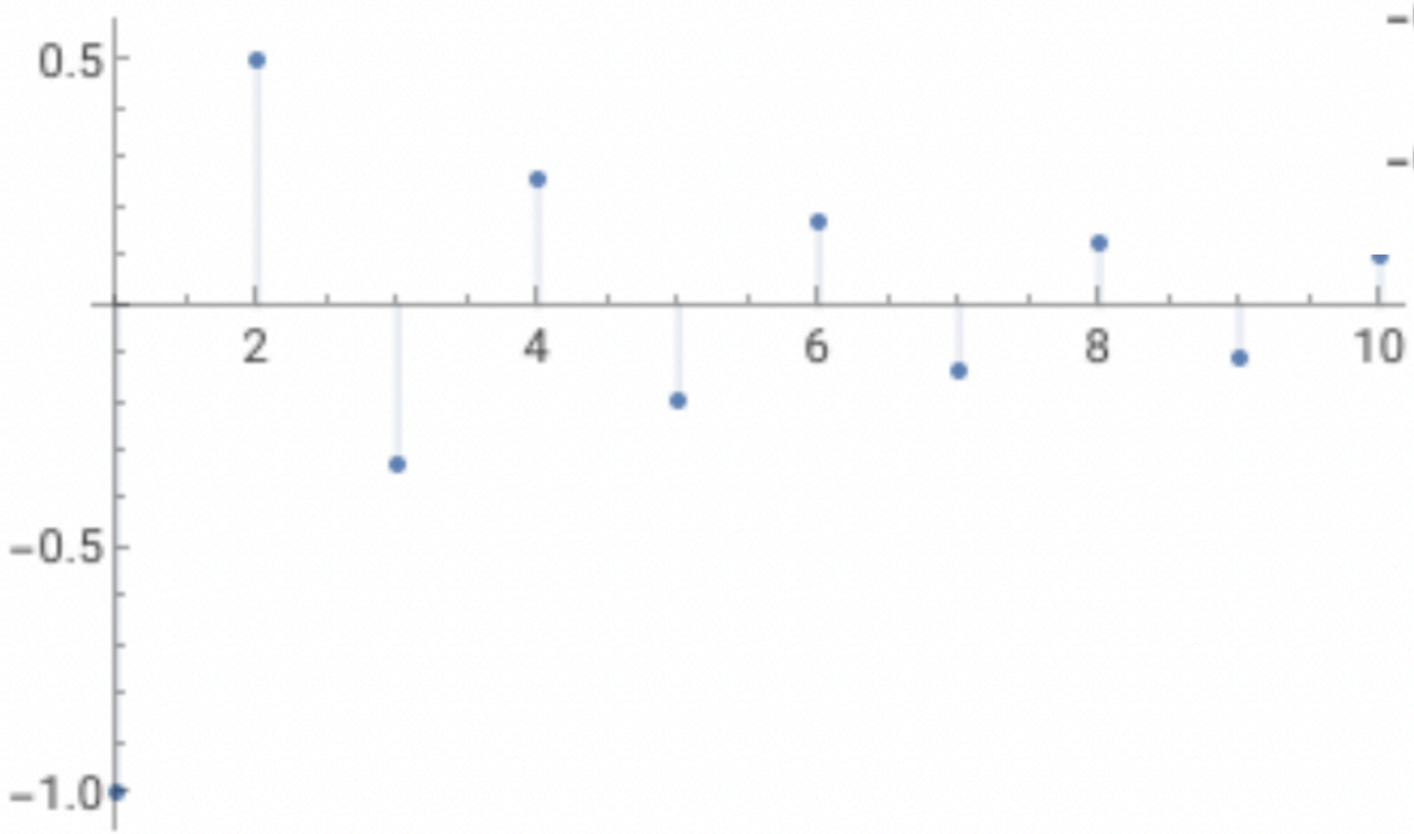
SEQUENCES

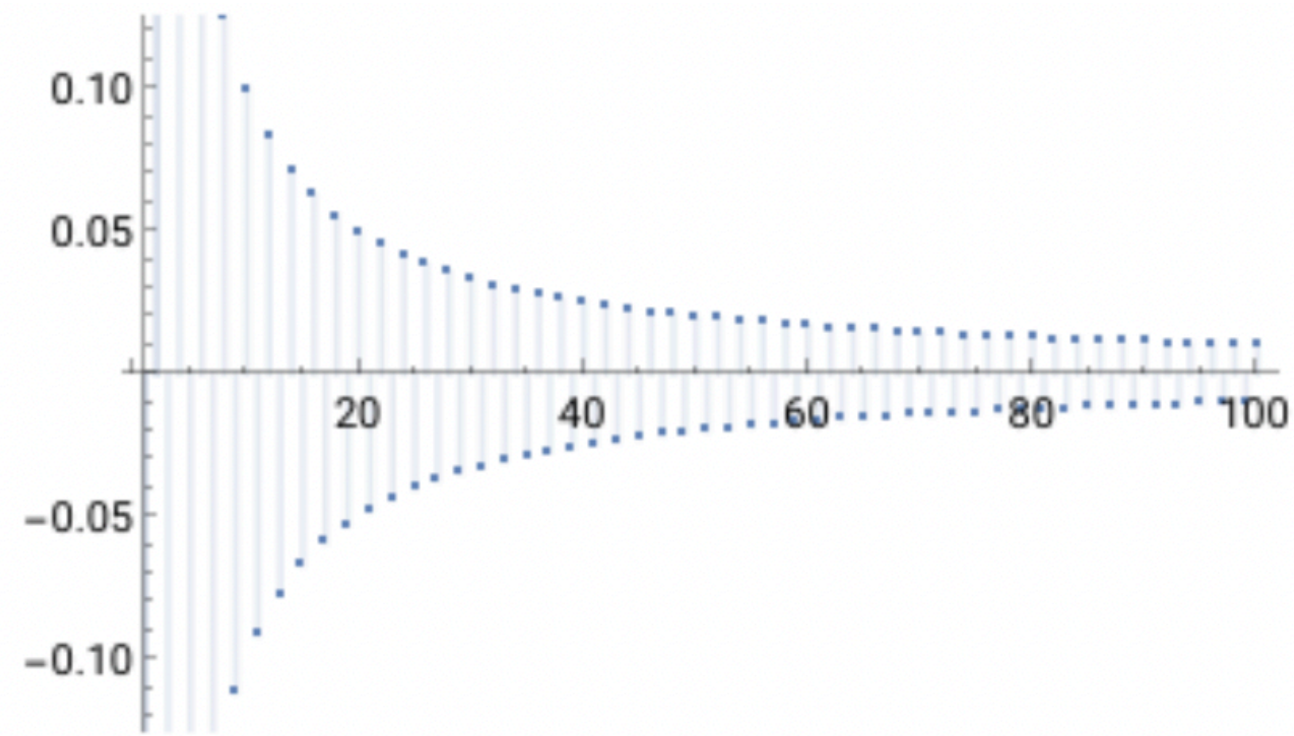
- $(\frac{(-1)^n}{n}) = -1,\ 1/2,\ -1/3,\ 1/4,\ \dots$
- Oscillating but "approach 0"

Number line

Decimal approximation

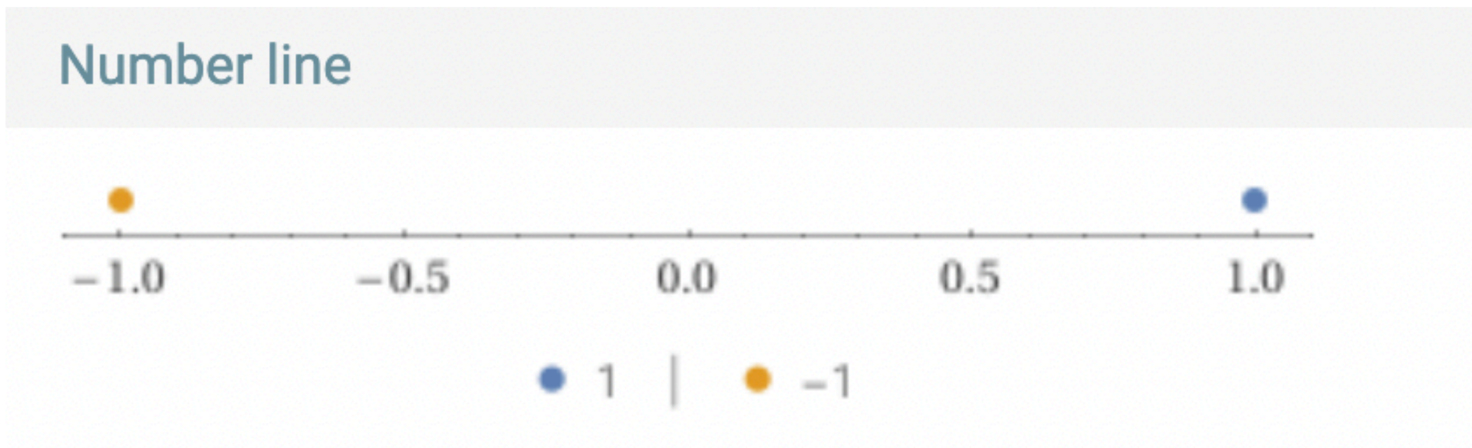
{−1, 0.5, −0.333333, 0.25, −0.2, 0.166667, −0.142857,

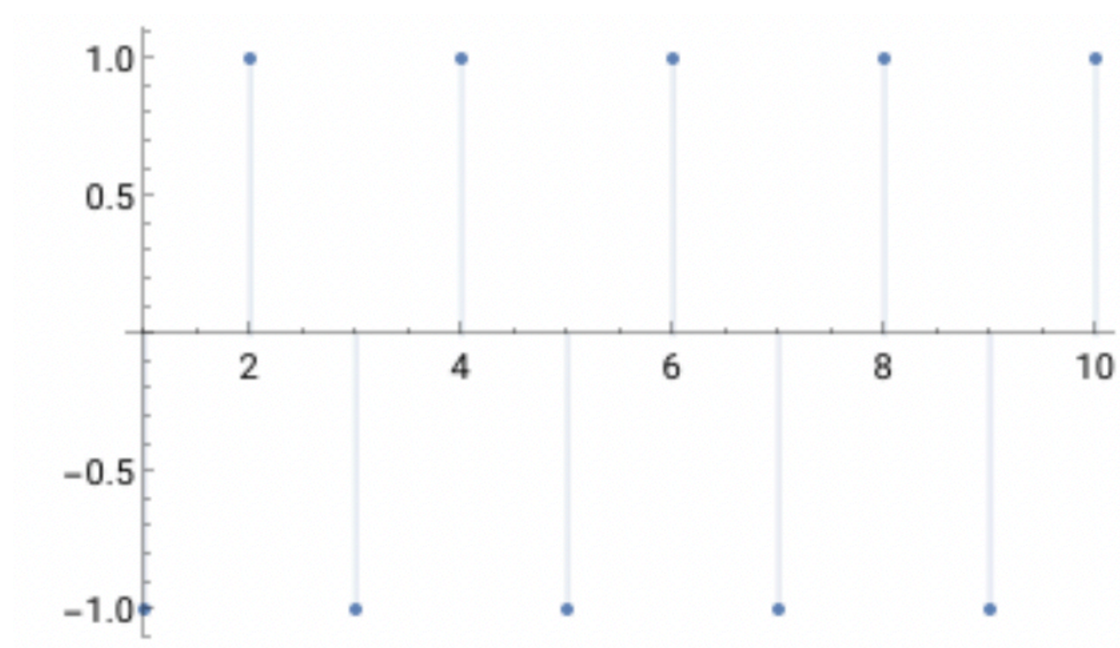




SEQUENCES

- $((-1)^n) = -1,\ 1,\ -1,\ 1,\ \ldots$
- Oscillating

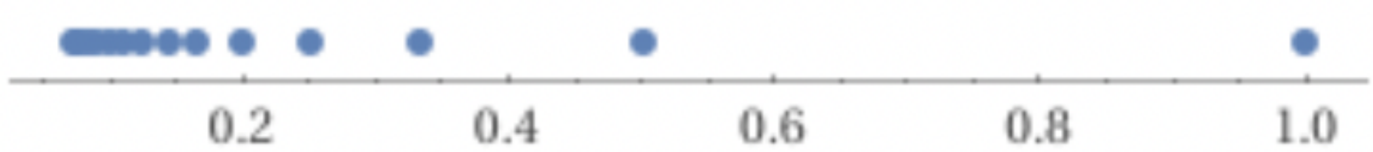




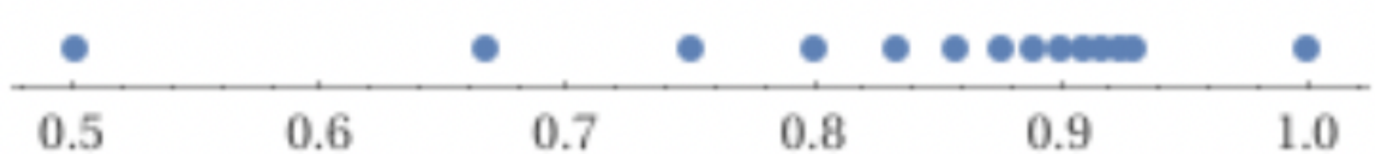
CONVERGENCE OF A SEQUENCE

- (x_n) is said to "converge to" the value ℓ if the terms of the sequence eventually "approach" ℓ (how to formalize "approach ℓ"?)
  - we can make x_i as close to ℓ as we want for all sufficiently large i
  - if such ℓ exists, the sequence is called convergent

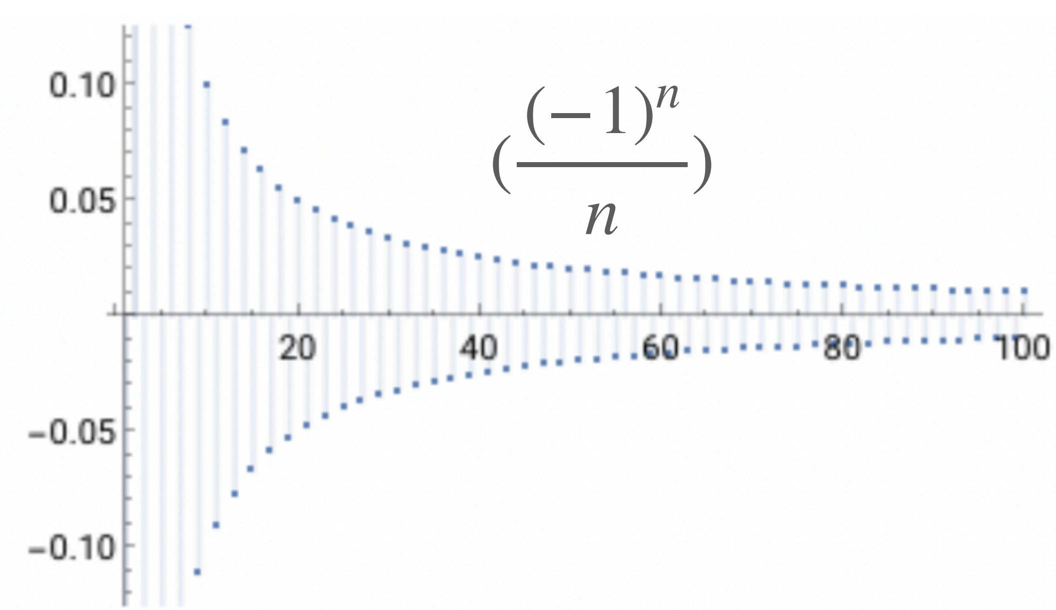
$(1/n)$



$(1 - \frac{1}{n})$



$(\frac{(-1)^n}{n})$



TOWARDS FORMAL DEFINITION OF CONVERGENCE

- For $\epsilon, \ell \in \mathbb{R}$ and $\epsilon > 0$ the ϵ-neighborhood of ℓ is

  $(\ell - \epsilon, \ell + \epsilon) := \{r \in \mathbb{R} : \ell - \epsilon < r < \ell + \epsilon\}$

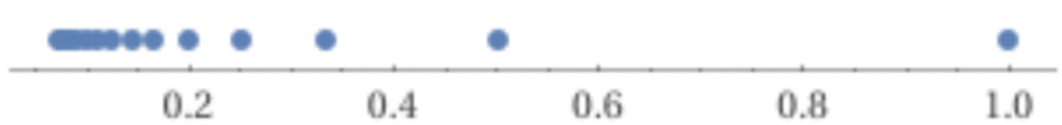
- Tail of a sequence (x_n) is $x_N,\ x_{N+1},\ x_{N+2}, \ldots$ for some $N \in \mathbb{N}$

- Example: Consider $(1/n)$
  - $1/10,\ 1/11,\ 1/12,\ \ldots$ is a tail
  - $1/107,\ 1/108,\ 1/109,\ \ldots$ is another tail

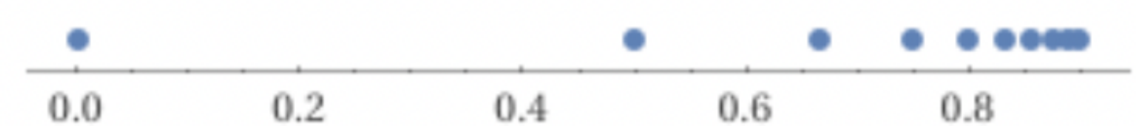
CONVERGENCE OF A SEQUENCE

- How can we define convergence formally?
- ϵ-neighborhood of ℓ is $(\ell - \epsilon, \ell + \epsilon) := \{r \in \mathbb{R} : \ell - \epsilon < r < \ell + \epsilon\}$
- Tail of a sequence (x_n) is $x_N, x_{N+1}, x_{N+2}, \ldots$ for some $N \in \mathbb{N}$

$(1/n)$



$(1 - 1/n)$



- (x_n) is said to converge to ℓ if every ϵ-neighborhood of ℓ contains a tail of (x_n)
  - $\lim_{n\to\infty} x_n = \ell$ (ℓ is the limit of (x_n))
  - $x_n \to \ell$ ((x_n) converges to ℓ)
- The limit of a sequence, if it exists, is unique. Why?