# SEQUENCES & LIMITS

Palakkad Math Circle | 23 Nov 2025

# SEQUENCES

- $\mathbb{N} = \{1, 2, 3, \ldots\}$
- Sequence: ordered list of real numbers $x_1, x_2, x_3, \ldots$ notation: $(x_n)$
  - $(1/n) = 1, 1/2, 1/3, 1/4, \ldots$ Decreasing
  - $(2n - 1) = 1, 3, 5, 7, \ldots$ Increasing
  - $(5) = 5, 5, 5, \ldots$ Constant sequence - both increasing and decreasing
  - $(1 - \frac{1}{n}) = 0, 1/2, 2/3, 3/4, \ldots$ Increasing
  - $(\frac{(-1)^n}{n}) = -1, 1/2, -1/3, 1/4, \ldots$ Oscillating
  - $((-1)^n) = -1, 1, -1, 1, \ldots$Oscillating
- $(x_n)$ is increasing if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ (monotone)
- $(x_n)$ is decreasing if $x_1 \geq x_2 \geq \ldots \geq x_n \geq \ldots$ (monotone)

# SEQUENCES & LIMITS

- What happens to the sequence $(x_n)$ as $n$ increases?
  - $(1/n) = 1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$ "approaches 0"
  - $(2n - 1) = 1,\ 3,\ 5,\ 7,\ \ldots$
  - $(5) = 5,\ 5,\ 5,\ \ldots$ constant
  - $(1 - \frac{1}{n}) = 0,\ 1/2,\ 2/3,\ 3/4,\ \ldots$"approaches 1"
  - $(\frac{(-1)^n}{n}) = -1,\ 1/2,\ -1/3,\ 1/4,\ \ldots$ oscillating but "approaches 0"
  - $((-1)^n) = -1,\ 1,\ -1,\ 1,\ \ldots$ oscillating
- How to formalize "approaches $\ell$"?

# TAILS & NEIGHBORHOOD

- Tail of a sequence $(x_n)$ is $x_N, x_{N+1}, x_{N+2}, \ldots$ for some $N \in \mathbb{N}$
  - $(1/n) = 1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$
    - $1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$ is a tail of $(1/n)$
    - $1/94,\ 1/95,\ 1/96,\ 1/97,\ \ldots$ is a tail of $(1/n)$
- Write 3 tails of $(1/n)$ and $(1 - \dfrac{1}{n})$
- $\epsilon$-neighborhood of $\ell$ is $(\ell - \epsilon, \ell + \epsilon) := \{r \in \mathbb{R} : \ell - \epsilon < r < \ell + \epsilon\}$
  - For $\ell = 2,\ \epsilon = 0.5,\ (1.5{,}2.5) := \{r \in \mathbb{R} : 1.5 < r < 2.5\}$
  - For $\ell = 1.5,\ \epsilon = 1,\ (0.5{,}2.5) := \{r \in \mathbb{R} : 0.5 < r < 2.5\}$
  - For $\ell = 0,\ \epsilon = 0.01,\ (-0.01{,}0.01) := \{r \in \mathbb{R} : -0.01 < r < 0.01\}$

# CONVERGENCE

- Definition: $(x_n)$ is said to converge to $\ell \in \mathbb{R}$ if every $\epsilon$-neighborhood of $\ell$ contains a tail of $(x_n)$ and we say $\ell$ is the limit of $(x_n)$ and write as $\lim_{n\to\infty} x_n = \ell$ or $x_n \to \ell$, we also say $(x_n)$ converges to $\ell$
- $(1/n) = 1,\ 1/2,\ 1/3,\ 1/4,\ \ldots,$ converges to 0
- $(5) = 5,\ 5,\ 5,\ \ldots$ converges to 5
- $(1 - \frac{1}{n}) = 0,\ 1/2,\ 2/3,\ 3/4,\ \ldots$ converges to 1
- $(\frac{(-1)^n}{n}) = -1,\ 1/2,\ -1/3,\ 1/4,\ \ldots$ converges to 0
- $(2n - 1) = 1,\ 3,\ 5,\ 7,\ \ldots$ does not converge?
- $((-1)^n) = -1,\ 1,\ -1,\ 1,\ \ldots$ does not converge?

# CALCULATING LIMITS

- Show that $1/n \to 0$
  - Take $\epsilon = 0.01$ and show that a tail is contained in $(-0.01, 0.01)$
    - We want $\dfrac{1}{n} < 0.01$ so take $n > 100$ and $1/101,\ 1/102,\ 1/103,\ \ldots$ is the required tail
  - Take $\epsilon = 0.001$ and show that a tail is contained in $(-0.001, 0.001)$
    - We want $\dfrac{1}{n} < 0.001$ so take $n > 1000$ and a required tail is $1/1001,\ 1/1002,\ 1/1003,\ \ldots$
  - Consider arbitrary $\epsilon > 0$
    - We want $\dfrac{1}{n} < \epsilon$ so take $n > 1/\epsilon$. Let $N$ be a natural number $> \dfrac{1}{\epsilon}$ and $1/N,\ 1/(N+1),\ 1/(N+2),\ \ldots$ is the required tail

# CALCULATING LIMITS

- Show that $1 - 1/n \to 1$
  - Consider arbitrary $\epsilon > 0$
    - We want $1 - \dfrac{1}{n} < \epsilon + 1$ and so take $n > 1/\epsilon$
    - Let $N$ be a natural number $> \dfrac{1}{\epsilon}$
    - $1 - 1/N,\ 1 - 1/(N+1),\ 1 - 1/(N+2),\ \ldots$ is the required tail
- Show that $(1) \to 1$
  - Consider arbitrary $\epsilon > 0$
  - We want $1 < \epsilon + 1$ which is true for all $n \geq 1$

# CALCULATING LIMITS

- When do you say $(x_n)$ does not converge to $\ell$?
  - $\exists \epsilon > 0$ such that the $\epsilon$-neighborhood of $\ell$ does not contain any tail


- Show that $1/n \nrightarrow 0.5$
  - Take $\epsilon = 0.1$ and observe that $(0.4, 0.6)$ has no tail of $(1/n)$
- Show that $(-1)^n \nrightarrow 1$
  - Take $\epsilon = 1$ and observe that $(0,2)$ has no tail of $(-1)^n$ as $-1 \notin (0,2)$
- To show $((-1)^n) \nrightarrow \ell$ for any $\ell$, note that no tail is in $(\ell - 0.5, \ell + 0.5)$
- Subsequences $(1)$ and $(-1)$ of $((-1)^n)$ converge to $1$ and $-1$ resp.

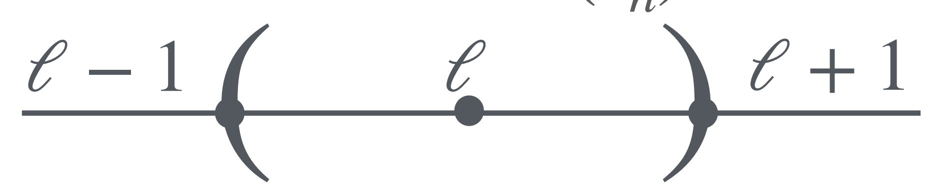
# UNIQUENESS OF LIMIT

- The limit of a sequence, if it exists, is unique
  - Suppose there are two different limits $\ell$, $\ell'$
  - Take $\epsilon > 0$ such that $(\ell - \epsilon, \ell + \epsilon)$ & $(\ell' - \epsilon, \ell' + \epsilon)$ have no common number


  - $(\ell - \epsilon, \ell + \epsilon)$ & $(\ell' - \epsilon, \ell' + \epsilon)$ contain a tail of $(x_n)$ - not possible

# CONVERGENCE, BOUNDEDNESS & MONOTONICITY

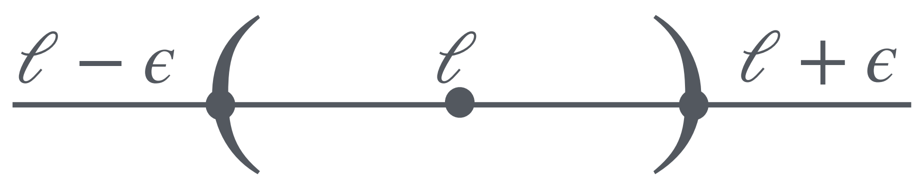
- When can monotone sequences converge?
  - $(1/n) = 1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$ decreasing, converges to 0
  - $(2n-1) = 1,\ 3,\ 5,\ 7,\ \ldots$ increasing, does not converge
  - $(5) = 5,\ 5,\ 5,\ \ldots$ both increasing and decreasing, converges to 5
  - $(1 - \frac{1}{n}) = 0,\ 1/2,\ 2/3,\ 3/4,\ \ldots$ increasing, converges to 1
- $(x_n)$ is bounded if there are numbers $A$, $B$ s.t for each $n$, $A \le x_n \le B$
- Every convergent sequence is bounded
  - Fix an $\epsilon$, say $\epsilon = 1$, let $x_i$ and $x_j$ be max and min element of $(x_n)$ that is outside $(\ell - \epsilon, \ell + \epsilon)$, if they exist




  - $\min\{x_j, \ell - \epsilon\}$ gives a lower bound and $\max\{x_i, \ell + \epsilon\}$ is an upper bound
- Can bounded sequences converge?

# CONVERGENCE, BOUNDEDNESS & MONOTONICITY

- Monotone Convergence Theorems
  - If $(x_n)$ is increasing and bounded then $(x_n)$ converges to its least upper bound (which exists by the completeness of $\mathbb{R}$)




$\ell$ is the lub

    - As $\ell$ is the least upper bound, $\ell - \epsilon$ is not an upper bound
    - There exists $x_N \in (\ell - \epsilon, \ell + \epsilon)$
    - As $(x_n)$ is increasing, its tail starting from $x_N$ is contained in $(\ell - \epsilon, \ell + \epsilon)$
  - If $(x_n)$ is decreasing and bounded then $(x_n)$ converges to its greatest lower bound (which exists by the completeness of $\mathbb{R}$)

# PROPERTIES OF LIMITS

- Consider sequence $(x_n)$ such that $x_n \geq 0$ for all $n$. Then $\lim_{n\to\infty} x_n$ (if it exists) $\geq 0$.
- Consider sequence $(x_n)$ such that $x_n > 0$ for every $n \in \mathbb{N}$. If $(x_n) \to \ell$ then $(1/x_n) \to 1/\ell$
- Consider sequences $(x_n)$ and $(y_n)$. If $(x_n) \to \ell$ and $(y_n) \to k$ then $(x_n + y_n) \to \ell + k$

# FIBONACCI SEQUENCE & GOLDEN RATIO

- Fibonacci sequence ($F_n$): 1, 1, 2, 3, 5, 8, 13, 21, 35, 56,…
  - $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$

- Fibonacci sequences appear in sunflowers, scales on pinecone, branching in trees, arrangement of leaves on a stem, the fruitlets of a pineapple, flower petals, nautilus shells, …

- ($F_n$) is an increasing sequence and ($F_n$) is not convergent

Images Source: Wikipedia, www.britannica.com, www.planetnatural.com and www.mathnasium.com

# FIBONACCI SEQUENCE & GOLDEN RATIO

- Fibonacci sequence $(F_n)$: 1, 1, 2, 3, 5, 8, 13, 21, 35, 56,…
  - $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$
- Let $(a_n)$ be a new sequence defined by $a_n = \frac{F_{n+1}}{F_n}$ for $n \geq 1$
- $(a_n)$ is bounded between 1 and 2
- $(a_{2n}) = a_2, a_4, a_6, a_8, \ldots$ is decreasing and converges to $\frac{1+\sqrt{5}}{2}$
- $(a_{2n-1}) = a_1, a_3, a_5, a_7, \ldots$ is increasing and converges to $\frac{1+\sqrt{5}}{2}$
- $(a_n)$ converges to $\frac{1+\sqrt{5}}{2}$ (golden ratio $\phi$)
- Two quantities $a > b > 0$ are in golden ratio if $\dfrac{a+b}{a} = \dfrac{a}{b}$